Rapid 3D finite-difference modelling for magnetotellurics based on Reduced Basis Method

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SUMMARY

Despite the fast development of the computer hardware and numerical simulation methods, the limited efficiency to solve the large-scale partial differential equations (PDE) in forward modelling is still the major obstacle to fast, real-time electromagnetic geophysical inversions. To address this issue, we introduce a reduced basis method to rapidly solve the PDE problem arise from the finite-difference discretization of curl-curl equations. The reduced basis method aims to project the full solution space of a PDE problem to its lower dimensional subspace, with a series of transformation basis. The reduced solution space can therefore be rapidly calculated by the spanning of a series of basis functions from the orthonormalization of a number of full solutions. The result from comparisons between the new method against its conventional counterpart with COMMEMI-3D synthetic test shows a promising speed-up of more than 100x. This may provide a new method to deal with challenges from near real-time simulations in industrial applications, or Bayesian inversions that requires millions of forward calculations.

Keywords: Reduced Basis, Electromagnetics, Forward Modelling, Finite Difference, Curl-Curl Equations

INTRODUCTION

In PDE based optimization problems, the speed of the forward modelling has always been the key to efficiently deal with large-scale problems, as with geophysical inversions. The rapid development of 3D modelling methods in the past decades have made it possible to perform complex large scale 3D EM inversions, which allow us to better understand the Earth. Recently, increasing applications of EM monitoring methods have emerged, like the fracturing monitoring in the oil/gas industry, or the groundwater monitoring in environmental engineering. All of these applications require repeated near real-time inversion results, which in turn calls for forward simulations with extremely high efficiency. However, even for modern clusters, conventional methods may still take hundreds to thousands seconds to solve the PDE problems, which fails to satisfy the requirement of the near real-time applications. Here we introduce a new reduced basis method (RBM) to rapidly solve the PDE problem arise from the finite-difference discretization of time-harmonic Maxwell equations, which are the foundation of most frequency domain EM problems. We show that our RBM can efficiently project the full solution space of a PDE problem to its lower dimensional subspace (Manassero et al., 2020), which drastically reduces the computational cost, while maintaining acceptable accuracy level.

METHODS

For most frequency-domain EM problems, the time-harmonic Maxwell's equations can be expressed as:

$$\begin{aligned} \nabla \times E &= -i\omega\mu H \\ \nabla \times H &= i\omega\varepsilon E + J + J^{ext}, \end{aligned} \tag{1}$$

where E and H are the electric and magnetic fields, ω is angular frequency, ε is the electric permittivity of the model domain, μ is the permeability, while $J = \sigma E$ denotes the electric conduction currents in the domain and J^{ext} stands for the external current forcing.
For quasi-static approximation (displacement currents are negligible) Equations (1) can be reduced to a second-order Curl-Curl problem based on electric fields. Without loss of generality, we can merge the internal and external current terms:

$$\nabla \times \nabla \times E + i\omega\mu\sigma E = J, \tag{2}$$

The equations (2) are often discretized (using e.g. finite element or finite difference methods) as linear equations of dimension N:

$$\mathrm{A}(\mu)\mathrm{x}(\mu) = \mathrm{b}(\mu), \tag{3}$$

where $\mathrm{A} \in \mathbb{R}^{\mathrm{N}\times\mathrm{N}}$, b and $\mathrm{x} \in \mathbb{R}^{\mathrm{N}}$. Note that for a given forward modelling problem, the system matrix A and the right-hand side b should both be functions of the model parameter $\boldsymbol{\mu}$ (in terms of physical and geometric property for a given parameter domain P). All the solutions x, which naturally also depend on the model parameter $\boldsymbol{\mu}$, form a Hilbert space (or solution space) V. Here we define the original PDE problem as the "full order" (FO) system, with which we define the residual as:

$$\mathrm{r}(\mathrm{x}\,;\mu) = \mathrm{b}(\mu) - \mathrm{A}(\mu)\mathrm{x}, \ \ \forall\, \mathrm{x} \in \mathbb{R}^{\mathrm{N}}, \tag{4}$$

The reduced basis method

For large scale 3D EM problems, solving the above FO system may entail significant computational costs, as the dimension of the problem (N) can be forbiddingly large to solve. Consider a "reduced order" (RO) system of dimension N_R that approximates (3):

$$\mathrm{A_R}(\mu)\mathrm{x_R}(\mu) = \mathrm{b_R}(\mu), \tag{5}$$

where $\mathrm{A_R}(\mu) \in \mathbb{R}^{\mathrm{N_R}\times\mathrm{N_R}}$, $\mathrm{b_R}(\mu)$ and $\mathrm{x_R}(\mu) \in \mathbb{R}^{\mathrm{N_R}}$. The problem can be greatly simplified, if we can find a projection, with which any $\mathrm{x}(\mu)$ in V can be approximated well enough (Quarteroni et al., 2016). By a linear combination of the RO solutions, the FO solution can be expressed in the form of:

$$\mathrm{x}(\mu) = \mathbb{V}\mathrm{x_R}(\mu), \tag{6}$$

where $\mathbb{V} \in \mathbb{R}^{\mathrm{N}\times\mathrm{N_R}}$, is a $\mu-$independent projection matrix, which maps the reduced solution space to the original solution space ($V_R \to V$). In other words, eq. (6) can be considered as the algebraic form of a Galerkin method over a subspace of dimension N_R from the N dimension space. The solution $\mathrm{x_R}$ of (5), can be determined by enforcing a suitable orthogonality criterion on the residual of the solution. We therefore have a FO residual estimation of:

$$\mathrm{r^R} = \mathrm{r}(\mathbb{V}\mathrm{x_R}\,;\mu), \tag{7}$$

using the RO solution from (5). Now if we can enforce the condition that the orthogonal projection of residual (7) onto the reduced solution space V_R is zero:

$$\mathbb{V}^{\mathrm{T}}\big(\mathrm{b}(\mu) - \mathrm{A}(\mu)\mathbb{V}\mathrm{x_R}(\mu)\big) = 0, \tag{8}$$

we obtain the RO problem (5) from the FO system. Then through this projection, we can map the reduced system from the FO system by:

$$\begin{aligned} \mathrm{A_R}(\mu) &= \mathbb{V}^{\mathrm{T}}\mathrm{A}(\mu)\mathbb{V}, \\ \mathrm{b_R}(\mu) &= \mathbb{V}^{\mathrm{T}}\mathrm{b}(\mu), \end{aligned} \tag{9}$$

which effectively construct a reduced subspace of from the original space (dimension $N \rightarrow N_R$).

The assembly of the reduced basis functions

To evaluate the reduced system (5), we still need to assemble the projection matrix $\mathbb{V}$, sometimes called the reduced basis function (RBF) beforehand. To this end, we start from a set of FO solutions (a.k.a. snapshots):

$$\{\mathrm{x}(\mu^1),\ \mathrm{x}(\mu^2), \dots,\ \mathrm{x}(\mu^{\mathrm{R}})\}, \tag{11}$$

with respect to a set of R model parameters. We can build a set of R functions (called "basis functions") by orthonormalize those snapshots

$$\{\nu^1,\ \nu^2, \dots,\ \nu^R\}, \tag{12}$$

regarding a suitable inner scaler product operation:

$$\left(\nu_j, \nu_k\right)_R = \delta_{jk}, where\ 1 \le j, k \le R \tag{13}$$

Then we can generate the reduced basis space by:

$$V_R = span\{\nu^1, \quad \nu^2, \dots, \quad \nu^R\}$$
$$= span\{x(\mu^1),\ x(\mu^2), \dots,\ x(\mu^R)\}, \tag{14}$$

which is nested (i.e., $V_{R-1} \subset V_R$). Since the reduced basis naturally also belongs to the original solution space V, we can expand the reduced basis function with respect to the original basis functions:

$$\nu_m = \sum_{i=1}^{N} \nu_m^{(i)} \varphi^i, 1 \le m \le N \tag{15}$$

where $\varphi = \{\varphi^1, \varphi^2, \dots, \varphi^N\}$ is the basis function of the original solution space V. Then the projection matrix can be assembled as:

$$(\mathbb{V})_{im} = \nu_m^{(i)}, 1 \le m \le N_R, 1 \le i \le N, \tag{16}$$

Exploiting the nested nature of V_R, we may use a greedy algorithm to recursively select a number of snapshots for the reduced basis space, by a certain optimal criterion. For example, we can add more snapshots, till the residual (7) satisfies the precision requirement.

In summary, the reduce basis method can be roughly divide into two stages:

1. the offline stage:
a) generate the original finite difference discretization of the curl-curl equations (3);
b) calculate a number of FO solution snapshots (11) using greed algorithm, and:
c) assemble the projection matrix (15) to build the reduced system;

2. the online stage:
a) build and solve the reduced system (5);
b) recover the approximate solution of the full system (6) and appraise the residual with (8);

SYNTHETIC EXAMPLES

To test the performance of our new RB method, we compare the modelling performance of the new method against that of its conventional finite difference counterpart, using the COMMEMI-3D synthetic model for magnetotellurics (Zhdanov et al., 1997; Fig. 1). The model domain is discretized into a 71 by 71 by 53 mesh, with a DoF of about 0.8 million. After the offline building stage of the reduced subspace (may take hours for personal computers), the DoFs of the approximate system can be reduced to merely a few hundred to thousand.

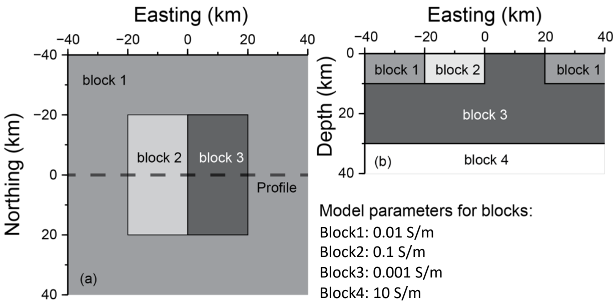

Figure 1: parameter setup of the COMMEMI-3D2 model (modified from Zhdanov et al., 1997)

To solve the full order linear system arises from the conventional FD method, we use an iterative QMR solver combined with divergence correction method.

As for the reduced system, we can use the direct method (Gaussian elimination with LU decomposition) as the DoF of the reduced system is small enough. As a result, the time for the forward problem reduces from about 105 s to less than 1s (Table 1) on a laptop computer with 8-core Apple M1 processor.

Table 1: efficiency comparison between the conventional full and reduced order systems

method	Mode	DoFs	Period	Walltime
Full Order	XY	826848	0.1s	103.20s
			100s	104.48 s
	YX	826848	0.1s	105.22 s
			100s	106.11 s
Reduced Order	XY	214	0.1s	0.93s
			100s	0.93s
	YX	228	0.1s	0.97s
			100s	0.98s

On the other hand, the computed electromagnetic fields from the RO system show almost identical results, when compared with its FO counterpart (Fig. 2), with acceptable precision with relative residual of < 10^{-5}.

CONCLUSIONS

We have developed a new reduced basis method to solve the time-harmonic Maxwell problem related to EM inversions. The efficiency of the new method allows us to rapidly perform approximate forward modelling calculations for moderate-scale 3D MT problems, which may provide a new way to improve the efficiency of electromagnetic inversions used with near real-time monitoring problems. It may also provide a rapid forward modelling method for Bayesian inversions, which may require millions of forward calculations in 3D cases.

ACKNOWLEDGMENTS

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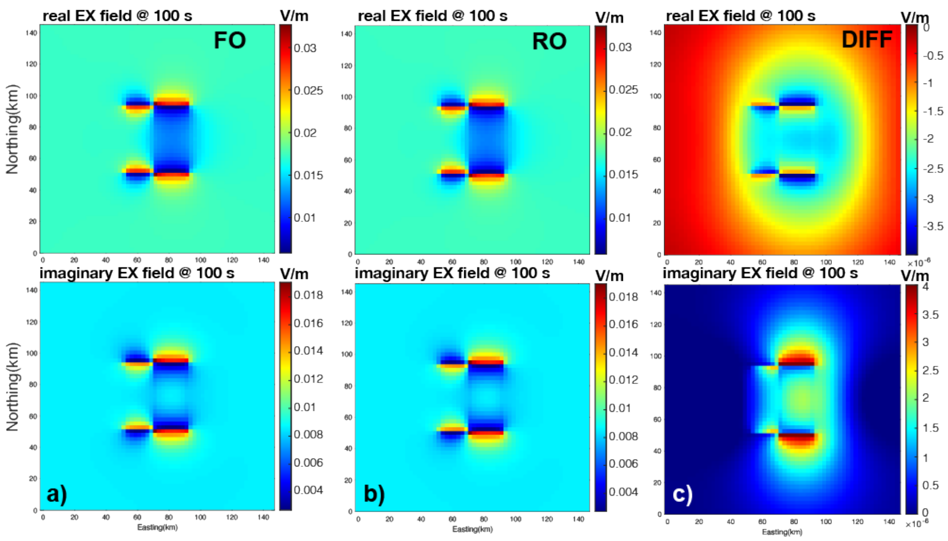

Figure 2: comparison of the XY polarization MT Ex field responses of the COMMEMI-3D2 model. Panels a), b) represent the result from full order and reduced order systems; c) shows difference between a) and b);

REFERENCES

Manassero, M. C., Afonso, J. C., Zyserman, F., Zlotnik, S., & Fomin, I. (2020) A reduced order approach for probabilistic inversions of 3-D magnetotelluric data I: general formulation. Geophysical Journal International, 223(3), 1837–1863.

Quarteroni, A., Manzoni, A., Negri, F. (2016) Reduced Basis Methods for Partial Differential Equation. Springer, Cham, 2016, 92:13-165.

Zhdanov, M. S., Varentsov, I. M., Weaver, J. T., Golubev, N. G., & Krylov, V. A. (1997). Methods for modelling electromagnetic fields Results from COMMEMI—the international project on the comparison of modelling methods for electromagnetic induction. Journal of Applied Geophysics, 37(3–4), 133–271.